MATH 140A Review: Mathematical Induction

Facts to Know:

To show that a statement holds for every $n = 1, 2, 3, \ldots$ follow the following steps of *mathematical induction*:

1. Identify the statement that needs to be proven for $n = 1, 2, 3, \ldots$.
2. **(basis step)** Prove that the statement holds for $n = 1$.
3. **(induction step)**
   - Assume that the statement holds for $n = k$.
   - Show that the statement holds for $n = k+1$.

If the basis and induction steps hold, then by mathematical induction we conclude that the statement holds for $n = 1, 2, \ldots$.

Example: Show that $4 \mid (7^n - 3^n)$ for $n = 1, 2, 3, \ldots$.

1. We need to show that

$$4 \mid (7^n - 3^n) \quad (*)$$

   for $n = 1, 2, \ldots$.

   $4 \mid B$ iff $B = 4 \cdot A$, $A \in \mathbb{Z}$.

2. **(basis step)** We need to show that

$$4 \mid (7^1 - 3^1).$$

$$7^1 - 3^1 = 4 \cdot 1$$

3. **(induction step)**
   - Assume

$$4 \mid (7^k - 3^k)$$

   - We need to show

$$4 \mid (7^{k+1} - 3^{k+1})$$

Proof. We need to show

$$4 \mid (7^n - 3^n) \qquad (*)$$

for $n = 1, 2, \ldots$.

(base step) We have that

$$7^1 - 3^1 = 4 = 4 \cdot 1.$$

Thus, $4 \mid 7^1 - 3^1$. Thus, $(*)$ holds for $n = 1$.

(induction step) Assume that

$$4 \mid (7^k - 3^k).$$

We need to show

$$4 \mid (7^{k+1} - 3^{k+1}).$$

Since $4 \mid (7^k - 3^k)$, then

$$7^k - 3^k = 4 \cdot q, \qquad (**)$$

where $q \in \mathbb{Z}$. We have

$$7^{k+1} - 3^{k+1} = 7(7^k) - 3^{k+1}. \qquad (***)$$

$(**)$ implies that

$$7^k = 4 \cdot q + 3^k.$$

Thus,

$$\begin{aligned}(***) &= 7(4 \cdot q + 3^k) - 3^{k+1} \\ &= 7 \cdot 4 \cdot q + 7 \cdot 3^k - 3 \cdot 3^k \\ &= 7 \cdot 4 \cdot q + 3^k(7-3)\end{aligned}$$

$$= 4\left(7 \cdot q + 3^k\right).$$

Thus,

$$4 \mid \left(7^{k+1} - 3^{k+1}\right).$$

Thus, (*) holds for $k+1$.

Hence, by mathematical induction, (*) holds for $n = 1, 2, \ldots$.

$\square$